Faculty of Science, Technology, Engineering and Mathematics
M337 Complex analysis

M337 Solutions to Specimen exam 1

There are alternative solutions to many of these questions. Any correct solution that is set out clearly is worth full marks.

Question 1

(a) $i^{17} = i^4 \times i^4 \times i^4 \times i^4 \times i = 1 \times i = i$ 2

(b) $\dfrac{1+i}{2-i} = \dfrac{1+i}{2-i} \times \dfrac{2+i}{2+i} = \dfrac{(2-1)+(2+1)i}{2^2+1^2} = \dfrac{1+3i}{5}$ 2

(c) $\sinh(i\pi/6) = i\sin(\pi/6) = i/2$ 2

(d) First observe that

$$\operatorname{Log}(-8i) = \log|-8i| + i\operatorname{Arg}(-8i) = \log 8 - i\pi/2.$$

Hence

$$\begin{aligned}(-8i)^{1/3} &= \exp\left(\tfrac{1}{3}\operatorname{Log}(-8i)\right)\\ &= \exp\left(\tfrac{1}{3}(\log 8 - i\pi/2)\right)\\ &= \exp(\log 2 - i\pi/6)\\ &= 2(\cos\pi/6 - i\sin\pi/6) = \sqrt{3} - i.\end{aligned}$$ 4

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Question 2

(a) (i) We choose the standard parametrisation $\gamma(t) = t - ti \quad (t \in [0,1])$ of Γ.
Then $\gamma'(t) = 1 - i$, so

$$\begin{aligned}\int_\Gamma \operatorname{Im} z\,dz &= \int_0^1 (-t)\times(1-i)\,dt\\ &= (-1+i)\left[\tfrac{1}{2}t^2\right]_0^1 = \tfrac{1}{2}(-1+i).\end{aligned}$$ 3

(ii) Using the Reverse Contour Theorem, we see that

$$\int_{\widetilde{\Gamma}} \operatorname{Re}(iz)\,dz = -\int_\Gamma \operatorname{Re}(iz)\,dz.$$

Let $z = x + iy$. Then $\operatorname{Re}(iz) = \operatorname{Re}(ix - y) = -y = -\operatorname{Im} z$. Hence, by part (a)(i),

$$\int_{\widetilde{\Gamma}} \operatorname{Re}(iz)\,dz = -\int_\Gamma (-\operatorname{Im} z)\,dz = \int_\Gamma \operatorname{Im} z\,dz = \tfrac{1}{2}(-1+i).$$ 2

(b) By the Triangle Inequality,

$$|\sinh z| = \left|\tfrac{1}{2}(e^z - e^{-z})\right| \le \tfrac{1}{2}(|e^z| + |e^{-z}|).$$

Let $z = x + iy$. Then $|e^z| = |e^{x+iy}| = |e^x||e^{iy}| = e^x$ and $|e^{-z}| = e^{-x}$. If z belongs to $C = \{z : |z| = 2\}$, then $-2 \le x \le 2$, so

$$|\sinh z| \le \tfrac{1}{2}(e^x + e^{-x}) = \cosh x \le \cosh 2.$$

Next, for $z \in C$, we can use the backwards form of the Triangle Inequality to give

$$|z^5 - 2| \ge |z^5| - 2 = |z|^5 - 2 = 32 - 2 = 30.$$

Thus, for $z \in C$,

$$\left|\frac{2\sinh z}{z^5 - 2}\right| \le \frac{2\cosh 2}{30} = \frac{\cosh 2}{15}.$$

Since the function $f(z) = (2\sinh z)/(z^5 - 2)$ is continuous on the circle C, which has length 4π, we can apply the Estimation Theorem to give

$$\left|\int_C \frac{2\sinh z}{z^5 - 2}\, dz\right| \le \frac{\cosh 2}{15} \times 4\pi = \frac{4\pi\cosh 2}{15}.$$ 5

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Question 3

(a) We have

$$f(z) = \frac{z}{3(z^2 - 1/3)(z^2 - 3)} = \frac{z}{3(z - 1/\sqrt{3})(z + 1/\sqrt{3})(z - \sqrt{3})(z + \sqrt{3})}.$$

Hence f has simple poles at $\pm 1/\sqrt{3}$ and $\pm\sqrt{3}$.

Of these poles, only those at $\pm 1/\sqrt{3}$ lie inside the unit circle.

By the Cover-up Rule,

$$\operatorname{Res}(f, 1/\sqrt{3}) = \frac{1/\sqrt{3}}{3 \times 2/\sqrt{3} \times (1/3 - 3)} = -\frac{1}{16}.$$

Since f is an odd function, we see from HB C1 1.1(a), p59, that

$$\operatorname{Res}(f, -1/\sqrt{3}) = \operatorname{Res}(f, 1/\sqrt{3}) = -\frac{1}{16}.$$ 5

(b) Using the strategy for evaluating real trigonometric integrals, we obtain

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{1 + 3\sin^2 t}\, dt &= \int_C \frac{1}{1 + 3((z - z^{-1})/(2i))^2} \times \frac{1}{iz}\, dz \\
&= \int_C \frac{1}{1 + 3((z - z^{-1})/(2i))^2} \times \frac{4iz}{(2iz)^2}\, dz \\
&= \int_C \frac{4iz}{-4z^2 + 3(z^2 - 1)^2}\, dz \\
&= \int_C \frac{4iz}{3z^4 - 10z^2 + 3}\, dz \\
&= \int_C \frac{4iz}{(3z^2 - 1)(z^2 - 3)}\, dz,
\end{aligned}$$

where C is the unit circle.

By applying the Residue Theorem with the residues of f at the poles $\pm 1/\sqrt{3}$ inside C found in part (a), we see that

$$\int_0^{2\pi} \frac{1}{1+3\sin^2 t}\,dt = 4i \times 2\pi i\big(\operatorname{Res}(f,1/\sqrt{3}) + \operatorname{Res}(f,-1/\sqrt{3})\big)$$
$$= -8\pi \times \left(-\frac{1}{8}\right) = \pi.$$ 5

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Question 4

(a) Let $f(z) = z^7 + 5z^3 + 7$.

Define $g(z) = z^7$. If $|z| = 2$, then, by the Triangle Inequality,

$$|f(z) - g(z)| = |5z^3 + 7| \le |5z^3| + 7 = 5 \times 2^3 + 7 = 47.$$

Also, for $|z| = 2$, we have $|g(z)| = |z^7| = 128$. Therefore

$$|f(z) - g(z)| < |g(z)|, \quad \text{for } |z| = 2.$$

Since f and g are analytic on the simply connected region $\mathbb{C}$, and $\{z : |z| = 2\}$ is a simple-closed contour in $\mathbb{C}$, we see from Rouché's Theorem that f has the same number of zeros as g inside $\{z : |z| = 2\}$, namely 7.

Next, if $|z| \le 1$, then, by the backwards form of the Triangle Inequality,

$$|f(z)| = |7 + 5z^3 + z^7| \ge 7 - |5z^3| - |z^7| \ge 7 - 5 - 1 = 1.$$

So f has no zeros in $\{z : |z| \le 1\}$.

Therefore f has 7 zeros inside the annulus $\{z : 1 < |z| < 2\}$. 8

(b) Since f is a polynomial function with real coefficients, it satisfies $\overline{f(z)} = f(\overline{z})$, for all $z \in \mathbb{C}$. Therefore the non-real zeros of f occur in complex conjugate pairs, by HB C2 2.8, p67. But there are an *odd* number of zeros in the annulus $\{z : 1 < |z| < 2\}$, so they cannot all be in complex conjugate pairs. It follows that at least one of the zeros must be a real number. 2

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Question 5

(a) The conjugate velocity function $\overline{q}(z) = z^2$ is analytic on $\mathbb{C}$, so q is the velocity function for an ideal flow on $\mathbb{C}$, by HB D1 1.15, p81. 1

(b) A complex potential function for the flow is

$$\Omega(z) = \frac{z^3}{3},$$

since this function is a primitive of $\overline{q}$ on $\mathbb{C}$. Writing $z = x + iy$, we see that

$$\Omega(z) = \tfrac{1}{3}(x+iy)^3 = \tfrac{1}{3}(x^3 + 3ix^2y - 3xy^2 - iy^3).$$

Hence a stream function for the flow is

$$\Psi(z) = \operatorname{Im}\Omega(z) = x^2y - \tfrac{1}{3}y^3.$$

The streamlines are given by $\Psi(z) = k$, for real constants k. The streamline through the point $e^{i\pi/3} = 1/2 + i\sqrt{3}/2$ satisfies

$$\frac{1}{4} \times \frac{\sqrt{3}}{2} - \frac{1}{3} \times \frac{3\sqrt{3}}{8} = 0,$$

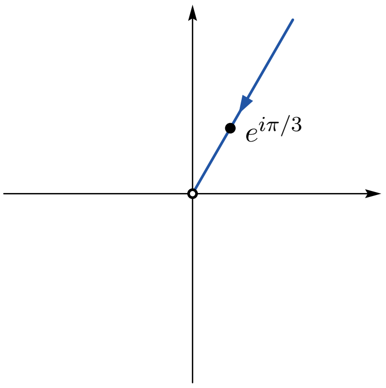
so $k = 0$. That gives $x^2 y = \frac{1}{3}y^3$, or, equivalently, $y(y^2 - 3x^2) = 0$. Factorising this, we obtain

$$y(y - \sqrt{3}x)(y + \sqrt{3}x) = 0.$$

This equation represents several streamlines, each separated by the degenerate streamline $z = 0$, a stagnation point for the flow. The streamline through $e^{i\pi/3}$ is the half-line

$$y = \sqrt{3}x, \quad x > 0.$$ 4

(c) Since $q(e^{i\pi/3}) = e^{-2i\pi/3}$, the direction of flow at $e^{i\pi/3}$ is $-2\pi/3$.



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(d) By HB D1 2.2, p82, we see that

$$\mathcal{F}_\Gamma = \operatorname{Im}(\Omega(i) - \Omega(-i)) = \operatorname{Im}(i^3/3 - (-i)^3/3) = \operatorname{Im}(-2i/3) = -2/3.$$ 2

Remark: The solutions to parts (b) and (c) can be shortened using polar coordinates $z = re^{i\theta}$ rather than $z = x + iy$. However, polar coordinates are not always suitable for questions of this type.

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Question 6

(a) By HB D2 2.1, p89, the iteration sequence

$$z_{n+1} = z_n(1 - z_n) = -z_n^2 + z_n, \quad n = 0, 1, 2, \ldots,$$

is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \ldots,$$

where

$$d = (-1) \times 0 + \tfrac{1}{2} \times 1 - \tfrac{1}{4} \times 1^2 = \tfrac{1}{4}.$$

The conjugating function is

$$h(z) = -z + \tfrac{1}{2}.$$

Hence

$$w_0 = h(z_0) = -\tfrac{1}{2} + \tfrac{1}{2} = 0.$$ 4

(b) (i) Let $c = \frac{1}{2}i$. Observe that

$$\left(8\left|\tfrac{1}{2}i\right|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\left(\tfrac{1}{2}i\right) = \left(2 - \tfrac{3}{2}\right)^2 + 0 = \tfrac{1}{4}.$$

Since $\frac{1}{4} < 3$, we see from HB D2 4.11(a), p92, that the function P_c has an attracting fixed point. Hence $\frac{1}{2}i \in M$, by HB D2 4.10, p92. 3

(ii) Let $c = 1 + \frac{1}{2}i$. Observe that

$$P_c(0) = 1 + \tfrac{1}{2}i,$$
$$P_c^2(0) = \left(1 + \tfrac{1}{2}i\right)^2 + \left(1 + \tfrac{1}{2}i\right) = \tfrac{7}{4} + \tfrac{3}{2}i.$$

So

$$|P_c^2(0)| = \tfrac{1}{4}\sqrt{7^2 + 6^2} = \tfrac{1}{4}\sqrt{85} > 2.$$

Hence $1 + \frac{1}{2}i \notin M$, by HB D2 4.6, p92. 3

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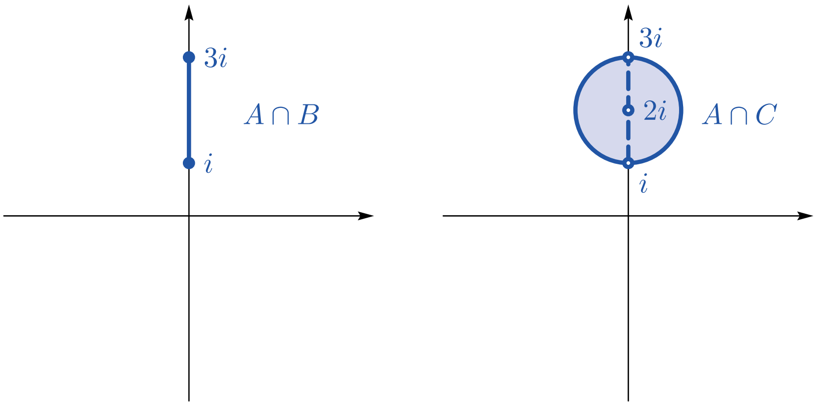
Question 7

(a) (i)



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(ii)
- The set A is closed and bounded, so it is compact. The function f is continuous on $\mathbb{C} - \{0\}$, so it is continuous on A. Hence f is bounded on A, by the Boundedness Theorem.
- Choose any point iy in B, where $y > 0$, and let $\theta = 1/y$. Then

  $$|f(iy)| = \left|e^{1/(iy)}\right| = \left|e^{-i/y}\right| = |e^{-i\theta}| = 1.$$

  Hence f is bounded on B.
- Observe that the point $1/n \in C$, for any $n \in \mathbb{N}$, and

  $$|f(1/n)| = |e^n| = e^n \to \infty \text{ as } n \to \infty.$$

  Hence f is not bounded on C. 6

(b) Let $z = x + iy$. Then

$$f(z) = \cos(x - iy) = \cos x \cos(iy) + \sin x \sin(iy) = \cos x \cosh y + i \sin x \sinh y.$$

Define

$$u(x,y) = \cos x \cosh y \quad \text{and} \quad v(x,y) = \sin x \sinh y.$$

Then $f(z) = u(x,y) + iv(x,y)$, and

$$\frac{\partial u}{\partial x}(x,y) = -\sin x \cosh y,$$

$$\frac{\partial u}{\partial y}(x,y) = \cos x \sinh y,$$

$$\frac{\partial v}{\partial x}(x,y) = \cos x \sinh y,$$

$$\frac{\partial v}{\partial y}(x,y) = \sin x \cosh y.$$

The first Cauchy–Riemann equation is

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \iff -\sin x \cosh y = \sin x \cosh y \iff \sin x \cosh y = 0.$$

Since $\cosh y \neq 0$, this equation is equivalent to the equation $\sin x = 0$, which has solutions $x = n\pi$, for $n \in \mathbb{Z}$.

The second Cauchy–Riemann equation is

$$\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) \iff \cos x \sinh y = -\cos x \sinh y \iff \cos x \sinh y = 0.$$

The solutions of this equation are $y = 0$ and $x = \left(n + \frac{1}{2}\right)\pi$, for $n \in \mathbb{Z}$.

Hence both the Cauchy–Riemann equations are satisfied if and only if $x = n\pi$, $n \in \mathbb{Z}$, and $y = 0$.

Since the partial derivatives exist and are continuous on $\mathbb{C}$, and the Cauchy–Riemann equations are satisfied at $z = n\pi$, $n \in \mathbb{Z}$, we see from the Cauchy–Riemann Converse Theorem that f is differentiable at these points with

$$\begin{aligned} f'(n\pi) &= \frac{\partial u}{\partial x}(n\pi, 0) + i\frac{\partial v}{\partial x}(n\pi, 0) \\ &= -\sin(n\pi)\cosh 0 + \cos(n\pi)\sinh 0 = 0. \end{aligned}$$

Since the Cauchy–Riemann equations are not satisfied at other points, the Cauchy–Riemann Theorem tells us that f is not differentiable at any other points of $\mathbb{C} - \{n\pi : n \in \mathbb{Z}\}$. 10

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Question 8

(a) (i) Let $w = z - 2$. Then $z = w + 2$, so

$$f(z) = \frac{4}{(w+2)^2 - 4} = \frac{4}{w^2 + 4w} = \frac{1}{w} \times \frac{1}{1 + w/4},$$

for $w \neq 0, -4$. If $0 < |z - 2| < 4$, then $0 < |w| < 4$, so $0 < |w/4| < 1$ and

$$\begin{aligned} f(z) &= \frac{1}{w} \times \frac{1}{1 + w/4} \\ &= \frac{1}{w} \times \left(1 - \frac{w}{4} + \left(\frac{w}{4}\right)^2 - \left(\frac{w}{4}\right)^3 + \cdots\right) \\ &= \frac{1}{w} - \frac{1}{4} + \frac{w}{4^2} - \frac{w^2}{4^3} + \cdots \\ &= \frac{1}{z-2} - \frac{1}{4} + \frac{z-2}{4^2} - \frac{(z-2)^2}{4^3} + \cdots \\ &= \frac{1}{z-2} - \frac{1}{4} + \frac{z-2}{16} - \frac{(z-2)^2}{64} + \cdots. \end{aligned}$$ 6

(ii) Let $w = z - 2$. Then

$$f(z) = \frac{4}{w^2 + 4w} = \frac{4}{w^2} \times \frac{1}{1 + 4/w},$$

for $w \neq 0, -4$. If $|z - 2| > 4$, then $|w| > 4$, so $|4/w| < 1$ and

$$\begin{aligned} f(z) &= \frac{4}{w^2} \times \frac{1}{1 + 4/w} \\ &= \frac{4}{w^2} \times \left(1 - \frac{4}{w} + \left(\frac{4}{w}\right)^2 - \left(\frac{4}{w}\right)^3 + \cdots\right) \\ &= \frac{4}{w^2} - \frac{4^2}{w^3} + \frac{4^3}{w^4} - \frac{4^4}{w^5} + \cdots \\ &= \frac{4}{(z-2)^2} - \frac{4^2}{(z-2)^3} + \frac{4^3}{(z-2)^4} - \frac{4^4}{(z-2)^5} + \cdots \\ &= \frac{4}{(z-2)^2} - \frac{16}{(z-2)^3} + \frac{64}{(z-2)^4} - \frac{256}{(z-2)^5} + \cdots. \end{aligned}$$ 6

(b) (i) For each $w \in \mathbb{C}$, we have

$$\cos w = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \cdots.$$

By substituting $w = 1/z^2$, for $z \neq 0$, and multiplying by z we obtain

$$\begin{aligned} z\cos(1/z^2) &= z\left(1 - \frac{(1/z^2)^2}{2!} + \frac{(1/z^2)^4}{4!} - \cdots\right) \\ &= z - \frac{1}{2!z^3} + \frac{1}{4!z^7} - \cdots \\ &= z - \frac{1}{2z^3} + \frac{1}{24z^7} - \cdots, \end{aligned}$$

for $z \neq 0$. 3

(ii) The Laurent series about 0 for g has infinitely many terms with negative powers. Hence 0 is an essential singularity of g, by HB B4 2.10(c), p57. 2

(iii) By the Casorati–Weierstrass Theorem with $\alpha = 0$, $\delta = \varepsilon = 1$ and $w = 1001i$, there is a complex number z with $0 < |z| < 1$ such that $|g(z) - 1001i| < 1$. Let $g(z) = u + iv$. Then

$$1001 - v \leq \sqrt{u^2 + (1001 - v)^2} = |g(z) - 1001i| < 1,$$

so $\operatorname{Im} g(z) = v > 1000$. 3

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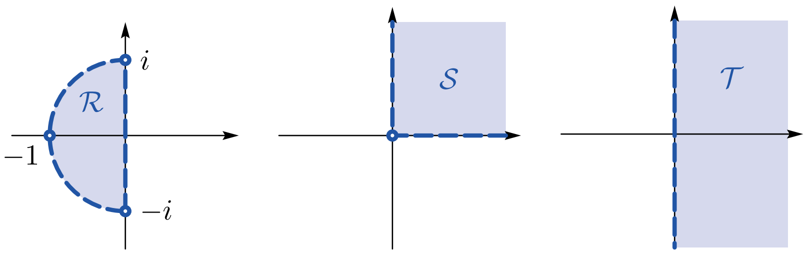
Question 9

(a)



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(b) Both $\mathcal{R}$ and $\mathcal{S}$ are lunes of angle $\pi/2$. The vertices of $\mathcal{R}$ are $-i$ and i, and the vertices of $\mathcal{S}$ are 0 and ∞. We can apply the strategy for mapping lunes to find a Möbius transformation f that maps $\mathcal{R}$ onto $\mathcal{S}$.

We choose f such that $f(-i) = 0$ and $f(i) = \infty$. Next we choose f to map the point 0 on the line segment from $-i$ to i (with $\mathcal{R}$ lying to the left) to the point 1 on the half-line from 0 to ∞ (with $\mathcal{S}$ lying to the left). Since

$$f(-i) = 0, \quad f(0) = 1, \quad f(i) = \infty,$$

we can apply the Explicit Formula for Möbius Transformations to see that

$$f(z) = \frac{(z - (-i))}{(z - i)} \frac{(0 - i)}{(0 - (-i))} = \frac{z + i}{-z + i}.$$

By the strategy, this transformation satisfies $f(\mathcal{R}) = \mathcal{S}$. Furthermore, because Möbius transformations are one-to-one and conformal on $\widehat{\mathbb{C}}$, we see that f is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$. 6

(c) We can write

$$\mathcal{S} = \{z : 0 < \operatorname{Arg} z < \pi/2\}.$$

The function $z \longmapsto z^2$ doubles arguments and it squares moduli, so it is a one-to-one mapping from $\mathcal{S}$ onto the upper half-plane $\{z : 0 < \operatorname{Arg} z < \pi\}$. We can then map this set onto $\mathcal{T}$ using a rotation $z \longmapsto -iz$ by $-\pi/2$ about 0. Hence $g(z) = -iz^2$ maps $\mathcal{S}$ onto $\mathcal{T}$.

This function g is a composition of one-to-one mappings, so it is a one-to-one mapping, and it is analytic because it is a polynomial function. Therefore it is a one-to-one conformal mapping from $\mathcal{S}$ onto $\mathcal{T}$. 4

(d) Since f is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$, and g is a one-to-one conformal mapping from $\mathcal{S}$ onto $\mathcal{T}$, the function $h = g \circ f$ is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{T}$. It has rule

$$h(z) = -i\left(\frac{z+i}{-z+i}\right)^2.$$

Next, using the inverse functions of the mappings $z \longmapsto z^2$ and $z \longmapsto -iz$, we see that

$$g^{-1}(z) = \sqrt{iz}.$$

Also,

$$f^{-1}(z) = \frac{iz-i}{z+1}.$$

Hence

$$h^{-1}(z) = f^{-1}(g^{-1}(z)) = \frac{i\sqrt{iz}-i}{\sqrt{iz}+1}.$$ 4

(e) The real line segment $L = (-1, 0)$ is the intersection of $\mathcal{R}$ with the extended real line. Now,

$$f(-1) = \frac{-1+i}{1+i} = i, \quad f(0) = 1, \quad f(\infty) = -1.$$

Hence f maps the extended real line to the unique generalised circle through i, 1 and -1, namely the unit circle. Then $g(z) = -iz^2$ maps the unit circle onto itself, so it follows that $h = g \circ f$ maps the real line segment L onto the intersection of $\mathcal{T}$ with the unit circle. That is,

$$h(L) = \{z : |z| = 1, \operatorname{Re} z > 0\}.$$ 3

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